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Aristotle and the Intuitionists

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Intuitionist mathematics has claimed a philosophy deriving from Kant. This paper aims to draw attention to significant similarities with a much older source, Aristotle. At the same time, the connection should not be over-stretched, given two millennia between them.

1. Introduction

In reading Aristotle I was struck by similarities between some of his views on mathematics, and those of the 20th century movement in mathematics known as *mathematical intuitionism*. So I want to say what these similarities are. I do NOT pretend that Aristotle was the first intuitionist, as there are definite differences. Still, the similarities are there, they are non-trivial, and it is instructive to bring them out. I will discuss three main themes: the law of excluded middle, counting, and the thesis that all infinity is potential infinity. I argue that both Aristotle and intuitionism can be seen, in their own distinctive ways, as taking similar positions on these themes. The intuitionists standardly attribute their intellectual heritage to Kant, but we will see that the lineage goes back much further. I am not the first to draw attention to this lineage, see for example Lear (1979–80).

2. Intuitionism

First, a brief introduction to intuitionism. This originated primarily with the great 20th century topologist Brouwer, and his student Heyting. Intuitionism is first of all a *constructivist* movement. Mathematics consists of constructions, and constructions are *effective* techniques. What constitutes constructiveness or effectiveness in mathematics is disputed territory between different styles of intuitionists, but it typically involves a step-by-step computation of some sort, such as could be done in principle by a computer, for example. For a constructivist to claim that a certain mathematical item exists requires that the item be displayed as the outcome of a computation. Mathematical techniques are by no means generally restricted to computer constructions, but intuitionists reject non-constructive techniques.

Examples will help. A well-known example of a non-constructive principle is the *Axiom of Choice*. This is a key principle in classical mathematics and is equivalent to many of the core theorems of modern analysis. But look at what it says. Take any set whose members are all sets; then *there exists* a function which picks out exactly one member from each of the sets. This is very simple and plausible sounding. But nonetheless it is rejected by intuitionists, who assent to no existential claim unless it is backed up by a construction (which the Axiom of Choice does not provide).

Here is pretty example of a non-constructive proof.

> **Theorem** There exist a pair of irrational numbers x and y such that x^y is rational.
> **Proof** Let $a = b = \sqrt{2}$, which is known to be irrational (Pythagoras). If a^b is rational then the theorem is true, since a and b are such a pair satisfying the theorem. So suppose a^b is irrational. Then $(a^b)^a = (a)^{ba} = (a)^{\sqrt{2}\sqrt{2}} = a^2 = 2$ which is rational. That is, a^b and a are such a pair of irrational numbers satisfying the theorem. QED

You should consult your own instincts on whether you feel that this proof is plausible. Classical mathematicians accept it. But the proof is not constructive because it does not construct *which* pair is rational, only argues that one *must* be.

Being contructivist, intuitionism is also *revisionist*. Significant chunks of classical mathematics, including the previous theorem, are rejected since they are not constructivist. Thus intuitionism is not just a philosophy of mathematics, but a new way of doing mathematics. Many mathematicians and logicians have since worked on the distinctively interesting mathematical problems thrown up by intuitionism, whether or not they believe in its advertised philosophy. Also, we have here a good example of philosophy driving mathematical discovery, not the other way around.

Intuitionism rejects the *Platonist* conception of mathematics, as postulating a mind-independent realm of abstract acausal Forms to be mathematical objects. Instead, all mathematical theorems come with an implied *date*. Mathematics isn't true *until* it has been created/constructed. Thus a mathematical assertion A is shorthand for "A has been established by a construction". Again, a negated proposition ¬A is shorthand for "The supposition that A has been constructed leads to absurdity" From this follows the characteristic intuitionist rejection of the Law of Excluded Middle, A∨¬A (that is, A-or-not-A), since it is clear that at a given time neither a construction for A, nor a construction showing the absurdity of A, might be available. Intuitionism also has a distinctive take on infinity. There are no real infinities, only potential infinities. We will look more closely at these themes in the following sections. We will compare Aristotle's views with intuitionism, and, where appropriate, we will also ask whether there are grounds for, or against, believing their views.

3. The Law of Excluded Middle (LEM)

It is well-known that Aristotle rejected LEM. But his reasons were rather different from Brouwer's. Consider any future contingent statement A (Aristotle's example was

"There will be a sea battle tomorrow"). If A were true now, then there could be no way that A would not happen tomorrow, thus A would be a necessary truth. Hence, if a statement A about the future is true now, it is necessarily true. Similarly, if ¬A is true now, then ¬A is necessarily true, and A is necessarily false . But A is contingent, that is neither A nor ¬A is necessarily true. Hence A isn't true, nor is ¬A true, so LEM fails (Aristotle, *De Interpretatione,* Ch. IX).

Aristotle reinforces his argument by pointing out that if A is true now, then if someone had said it 10,000 years ago it would have been true *then* that there will be a sea battle on the day after today's date (and we don't think we can change the past). He also uses future potentiality as an example of future contingency: we think that our arm has the potentiality to be cut off, so that is possible. It is also possible that it doesn't get cut off — the mere fact that it is still on now shows this.

It is commonplace to argue against Aristotle on this point, that the truth *now* of A could hardly guarantee the *logical* necessity of A. But I think that this is to underestimate A's argument. We all take it that there is a difference between the past and the future. We think we can change the future; but the past, having happened, cannot be changed. So we can reformulate A's argument as follows (still very much in his spirit). If A is true at a time ahead of the day of the battle, then it cannot be made false thereafter. Similarly if ¬A is true at a time ahead of the nominated day, then it cannot be made false thereafter. But we think that, before the day, we can make A true or make A false, and similarly for ¬A. Hence if A is contingent, then it is not true ahead of time, nor is ¬A true ahead of time.

Aristotle's argument contrasts with intuitionism, which rejects LEM but takes a rather different approach to doing so. The statement A∨¬A, means that either A has been established by a construction, or an absurdity follows from the supposition of a construction of A. Clearly, both of these might fail.

The intuitionists' argument relied on their dating of mathematical propositions: clearly A might not be constructed at a time, and ¬A might not be constructed at that time (Korner, 1960, Ch. VI). But there are two ways in which the argument can proceed from weaker premisses, and yet arrive at the same conclusion.

First, the rejection of LEM is not dependent on *actual* dates. Instead, say that a mathematical proposition is true if it is constructed at *some* time (or, at some point in spacetime), and say that it is false if at some time it constructively leads to absurdity. Then equally LEM can fail.

Second, we might dispense altogether both with constructability and with times. Say only that a mathematical proposition is true if it is *provable*, and false if it is provably false. This is a reasonable position, which has the merit of following the intuitionists in linking the truth of mathematics to its epistemology. The notion of "provable" is left open, cut loose from constructions, thus allowing a variety of options depending on one's standards for proof. These standards might include simple informal mathematical proof, the kind they do in classical mathematics departments. But it must still surely be allowed that neither A nor ¬A might be provable. The existence of reasonable incomplete theories is commonplace since Godel's incompleteness theorems for arithmetic. Even informal mathematical proof might be incomplete like this.

There is however a subtle problem for intuitionism here. Do we not think that mathematical theorems are proved "by historical accident", in the sense that the comet *might* have wiped out all life instead of just the dinosaurs? In that possible universe, all the mathematics we take to be true would not have been proved, and so not be true. This means that our true mathematics fails to be true in at least one other possible world. Is mathematics so contingent? This seems hard to accept.

I will push this argument a little further in the final section. To sum up this section: both Aristotle and Brouwer reject Excluded Middle, but for rather different reasons. It is clear that both of their arguments should not be underestimated. On the other hand, we can also see that there are *prima facie* problems for both positions. We will return to such problems in later sections.

4. Counting

Intuitionism is distinctive in that it places the intuitions of the apprehending mathematical mind at the forefront of mathematics. That mathematics should have a plausible epistemology is crucial: Platonism runs precisely into the problem of being unable to account plausibly for our knowledge of mathematics, because it postulates mathematical objects as abstract or acausal. This leaves an explanatory gap concerning how our minds could ever be engaged with them, in order to know them. Contrast this with vision, which is uncontroversial as a source of knowledge because it is a reasonable causal mechanism.

Brouwer postulated two fundamental intuitions, the *first and second acts of intuitionism*. The *first* act is "the falling apart of a life moment" which establishes the existence of *twoness*, and by iteration, *threeness, fourness,...* and so the *natural numbers* and the *integers*, of arbitrary size. This is *counting*. The *second* act is the intuition of *arbitrarily extended sequences of free mathematical choices*, together with the apprehension that some of these choices might cease to be free, when a *mathematical species* (or type) is apprehended. The ability to continue a series of choices indefinitely is what accounts for how an arbitrarily extended series of terms can be constructed, which arbitrarily closely approaches a *real number*.

According to Copleston (1962:64–66), Aristotle says that the consciousness of *time* is the consciousness of *plurality* "Nowness attached to different statements". This is strikingly like Brouwer's first act of intuitionism, wherein the natural numbers come to our awareness by consciousness falling apart into twoness, which is ultimately difference, that is non-identity.

Both Aristotle and Brouwer are entirely correct here, I would say. Counting is a primary intuition for mathematics, and indeed to deal with the practical problems of ordinary life. The first act of intuitionism, the falling apart of a life moment, is the apprehension of twoness, or disidentity. It was Frege who taught us how to count from this basis, without having to appeal to actually existing numbers. "There are two marbles in the tin" is "There is an x and a y with $x \neq y$ and x and y are marbles and in the tin." This is by now routinely included in introductory logic courses. It asserts the existence of marbles, not numbers and especially not Platonic Forms.

In passing, we can ask for more precise details, especially on whether this twoness is disidentity over space (at the same time), or twoness as disidentity over time (in the same place), as grasped by memory. The best answer would seem to be that the former suits Brouwer's counting, the latter suits Aristotle's awareness of time.

5. Potential and Actual Infinity

Both Aristotle and the intuitionists deny *actual* infinity, allowing only *potential* infinity. What can this mean?

Intuitively, a *finite* collection or set of things, is a set whose members can be counted using the discrete numbers 0,1,2... The counting in this case is "in principle" counting, since the counting might involve a number so huge that it couldn't realistically be counted up to before the heat-death of the universe. In passing, there exists an alternative definition of finitude, due to Dedekind: a finite set is one which cannot be placed in one-one correspondence with any of its proper subsets. Consider for example the two-element set {0,1}. Its proper subsets are {0}, {1}, and the null set. None of these can be placed in 1–1 correspondence with {0,1}, so the latter set is Dedekind-finite. These two definitions of finitude can be proved equivalent if the Axiom of Choice is assumed. Of course we have seen that intuitionists deny the Axiom of Choice, though most accept it. But there is no reason for us to make this assumption, as we can take the former, intuitive idea.

Hence, an infinite collection is one which cannot be numbered in this way. For example, the natural numbers {0,1,2, ...} are an infinite collection, as there is no way to start counting and count all its members, some will always be left out. In passing, we can see a use for Dedekind's alternative definition: the natural numbers are Dedekind-infinite because there is a 1–1 correspondence between the set of natural numbers and a proper subset thereof, the even natural numbers {0,2,4, ...}.

Someone who believes in actually infinite sets, then, is someone who holds that there are sets whose members cannot be counted. We must be careful not to "double count", or we could be counting the same things in a finite set over and over. We must therefore stipulate that an infinite set is a set whose members are distinct from one another and do not overlap, but is not finite, that is cannot be counted.

Both Aristotle and Brouwer denied infinite collections in this sense. This is often stated by saying that they denied actual infinity. The qualifier "actual" functions as a contrast with "potential", because Aristotle and Brouwer allow infinity in a weaker sense, usually referred to as "potential infinity". Both allow that the things to be counted might be such that if you started to count, *you could always count one more.* One way to say this, is to say that for any finite set there exists another (larger) set, one which must be counted by a higher number. Why is this not equivalent to actual infinity? Because Aristotle and Brouwer are not allowing the existence of an infinite set. They allow that arbitrarily large finite sets can exist, but a "completed infinity", an infinite set of simultaneously-existing, distinct, non-overlapping things, does not exist. This account of actual infinitude can be found in Trifogli (2000:12), who refers it to Bostock.

In what sense is potential infinity potential then? Generally, statements of potentiality are statements of dispositionality, and thus conditionality. In contrast, statements of actuality are statements of existence. Actual infinities involve infinite numbers of simultaneously-existing things, potential infinities involve the irreducibly-modal *you can always go one more*, without necessarily the existence of an actual infinity of things to explain that fact.

Aristotle distinguishes two kinds of (potential) infinity: *additive* (as in being able to add more) and *divisible* (as in being able to divide further). These correspond to the first and second acts of intuitionism. Number (counting numbers) are said to be additively infinite, body is said to be divisibly infinite, and time is said to be both. Time has no beginning and no end, and so it might seem that at least the infinity of past times is actual. But Aristotle replies ingeniously that this does not make time actually infinite, because the many times don't co-exist (*Physics,* 204b, 7–10).

Korner (1960:146) describes intuitionism as being a "moderate finitism", which allows a kind of surrogate infinity, that is potential infinity. According to Copleston, for Aristotle infinity exists *in the form of* potential infinity. These sound very like each other. However, it is worth foreshadowing the objection looming for both is that it is all well and good to distinguish additive and divisible infinities, but the prior question is how good an explanatory tool is potentiality? In what sense is potential infinity infinite, then? And, for that matter, what makes it potential? What are the truth makers of potentiality? Until these questions are answered, potentiality cannot be regarded as a useful explanatory concept. We return to these points in a later section.

Indeed, the case against *mere* potential infinity, potential infinity without actual infinity, is even stronger, as I will argue in the final section that a strong case for actual infinity can be made out.

6. The Continuum

While not strictly supplying us with a fourth similarity between Aristotle and Brouwer, the continuum is worth passing notice. Aristotle was aware of the difference between two sorts of collections, a sequence of discretely ordered things, and the continuum. His prevailing view of the difference was that in a discrete order everything has unique neighbours, while in the continuum there were no unique neighbours. Two millennia later we refer to the latter sort of order as *dense*, that is to say: *between any two elements there is a third.* This is a kind of infinite divisibility, in the sense that whenever the continuum is divided into distinct (non-punctate) parts, the resulting parts are in turn divisible further. Bostock (1991:186) argues that this is the most plausible interpretation of *Physics VI*.

We also know, what Aristotle could not be expected to know, that there is a distinction between two sorts of dense order, respectively the rational numbers and the real numbers. The rationals (the fractions) are embedded in the reals but omit those reals such as √2 which are non-fractions. The reals are characterised in classical mathematics by the further special property that *Dedekind cuts are of two kinds only*. This property is difficult to explain easily: suffice it for our purposes to say that it is a property of infinite sequences which fails for the rational numbers and so is standardly used to characterise the complete set of the reals. There are a number of equivalent ways to state it, for example that *the reals have all their limit points while the rationals do not* (for a thorough discussion of these concepts in the context of Aristotle see Bostock, 1991, especially 185–7).

Brouwer was well aware of the difference in non-intuitionist mathematics between the rationals and the reals. But for the intuitionist, continuity has to be dealt with constructively. Intuitionist "continuity" is not, as it is classically, the existence of all limit points. It is, instead, the ability, in a construction, to arbitrarily closely approach what would classically be called a real-number limit. But we cannot constructively tell the difference between a rational number (ultimately repeating decimal) and an irrational number (ultimately non-repeating decimal), because a construction to tell the difference would only go a finite distance. The apprehension of a real number, the second act of intuitionism, is the apprehension of a free choice sequence (an infinite Cauchy sequence converging on the number). However, one never apprehends the sequence in its entirety, it emerges dynamically. At any stage it can thus be indeterminate whether two such sequences are different (whether two real numbers are disidentical). The reals so considered form the intuitionist continuum.

Brouwer's views on the difference between rational and real without postulating limits, which take a real infinity to be reached, make for considerable complications, and take us far from Aristotle; so we will not take them up here.

7. Two Thought Experiments

Now I want to conclude by pursuing an objection to Aristotle foreshadowed in Section 5. Ross (1923:85) tells us that, for Aristotle, in the continuum there are no actually existing parts, only potential parts. It seems to me that he has to say this, or the following argument makes for difficulties for his denial of actual infinity. Indeed, it makes for difficulties anyway.

We perform a thought experiment. Recall that in Section 5 we adopted, from Trifogli, that actual infinity requires a non-finite collection of simultaneously co-existing things. We focus here in the simultaneous existence of such things. Imagine that I take hold of your hand, and with pliers I rapidly pull out your thumbnail. My question is: did your thumbnail exist before I separated it from the rest of your hand? The natural reply is: yes. That is not in accord with Aristotle according to Ross, but it is a reasonable insight nonetheless. It suggests strongly that the parts of a continuum exist prior to their being physically separated. But now notice that a Zeno-like argument can be mounted on the rest of the hand: separate half of what is left, then as we have just seen that part too existed before the separation. Separate half of what was left, ...*ad infinitum*. We thus have an infinite collection of co-existing parts of the thing, having no overlap with one another. But this is the definition of an actual infinity, as adopted from Trifogli. Conclusion: actual infinities exist. Even better, we can avoid a problem for Aristotle at the end of Chapter 5: the problem of the truthmakers for potentiality. The actual infinity of non-overlapping parts supplies us with the truthmakers for the statements about potential infinity: you can always make a bigger set *because* an infinite set of counting-numbers exists. But the cost for Aristotle is great, actual infinity. (However, for a more sympathetic account of Aristotle on these points, see White [1992], who is also more luke-warm on the similarities with intuitionism than I have been.)

Aristotle presumably has to respond to this thought experiment as Ross says he does, by saying that the parts don't exist before separation, but this is to deny the natural reply to the question I asked.

This has been a thought experiment about infinity by division. Can a similar argument be mounted against Aristotle to show that there is actual infinity by addition? I don't think so. Imagine a finite discrete co-existing collection of stones. One more can be added, and we still have a finite discrete collection. Can one more be added *no matter how many we have*? I see no reason to believe so. We would need to be assured ahead of time that the extra stone to be added already existed. But if we have got to the end of the universe and run out of stones, we do not have that assurance. Contrast with division, where from the previous thought experiment we do seem to have that assurance. Conclusion: Aristotle's denial of actual additive infinities resists refutation in a way that his denial of actual infinity by division does not.

Conclusion

To sum up: we have seen three definite similarities between Aristotle and Brouwer: excluded middle, counting, and the denial of actual infinity accompanied by support for potential infinity. We must be cautious about attributing more similarities than this, however, as their views elsewhere are not quite so similar on closer inspection, such as on the continuum. We also saw that there are interesting philosophical issues arising about the truth of these views, for instance that actual infinity is supported by a plausible argument.

Bibliography

Aristotle

Aristotle, *De Interpretatione, Physics,* in McKeon (ed) *The Basic Works of Aristotle*. New York: Random House, 1941 (reprinted from the Oxford edition, Ross [ed.] 1931).

Bostock, 1991

David Bostock, "Aristotle on Continuity in *Physics VI"*. In *Aristotle's Physics*, ed. L. Judson: 179–204. Oxford: The Clarendon Press.

Copleston, 1962

Frederick Copleston, *A History of Philosophy, vol. 1, part 2 Greece and Rome.* Westminster, Maryland: The Newman Press.

Korner, 1960

S. Korner, *The Philosophy of Mathematics*. New York: Dover.

Lear, 1979–80

Jonathan Lear, "Aristotelian Infinity". In *Proceedings of the Aristotelian Society*, New Series, 80:187–210.

Ross, 1923

D. Ross, *Aristotle.* London: Methuen.

Trifogli, 2000

Cecilia Trifogli, *Oxford Physics in the Thirteenth Century (ca. 1250–1270): Motion, Infinity, Place and Time.* Leiden: Brill.

White, 1992

M.J. White, "Aristotle and the Mathematicians, Ancient and Modern", *The Continuous and the Discrete* 4:133–188. Oxford: Clarendon Press.